DEPARTMENT OF MATHEMATICS COMPUTER SCIENCE & ENGINEERING TECHNOLOGY

Introducing Hypergraphs to Early College Students

by

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ABSTRACT

The growing importance of computational science and informatics has given rise to complex systems which are often modeled using hypergraphs; these are generalizations of graphs which are used in modeling phenomena in cyberspace, relations in information systems, social networks, etc. In this research, we explore several classes and families of hypergraphs. Two classes are considered, namely, linear and nonlinear. The former is well-studied and yet the later is barely known. We explore and classify various families of each class through the notions of linearity, uniformity, being balanced and semi-balanced along with their cyclic natures. After defining and proving some necessary conditions on the existence of some of these hypergraphs, we introduce several activities and applied problems along with their solutions to engage early college students on hypergraphs.

DEDICATION

This thesis is dedicated to my parents Melinda and Nicholas Vacca. My parents have always inspired me to do my best in anything that I take on and to never give up. They have always supported my decisions when it came to my education and what path I wanted to follow in life. I would also like to dedicate this thesis to my close friends. We have always encouraged each other to further our education and strive for the best. Without my parents and friends, I would not be where I am today. With that said, I am forever grateful to them.

ACKNOWLEDGEMENT

I would like to start by acknowledging the first teacher that encouraged me to become a mathematics teacher, Mr. Kevin Argo. I had the honor of being in one of Mr. Argo's mathematics classes at College of the Albemarle and learned what it meant to be a great teacher. It was then I decided that when I transferred to Elizabeth City State University that I would major in mathematics and go back to teach at the community college level. I would also like to acknowledge Elizabeth City State University as a whole. Anytime I needed assistance with my loans or had questions, the staff was some of the nicest people I have ever spoken with. It has also been a blessing to have such a great institution so close to home and be able to earn not only my bachelor's degree from here, but now my master's degree. In addition, I would like to acknowledge the mathematics department for providing us with the best professors we could ever ask for. I had the honor of working with Dr. Julian Allagan on my thesis. His knowledge on the topic helped me in many ways on this thesis and I have learned so much from him. Dr. Allagan has also been a great professor in the undergraduate and graduate courses I have taken with him. I would like to thank Dr. Kenneth L. Jones for being an incredible professor and learning to see the brilliance in his "beautiful mathematics". In the very first course I took with Dr. Jones, he was already encouraging us to further our education in their graduate program. I would like to thank Dr. Dipendra Sengupta, as I took the majority of my courses under him, I have truly learned so much. Dr. Sengupta has always pushed me to do better and it has shown over the years. Lastly, I would like to thank Dr. Talukder, as he was always ready to help me and his other students when we needed extra help learning some material. Thank you again to all of my professors, I appreciate everything you all have taught me.

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Chapter 1 Introduction

In this work, we researched deeper into the topic of hypergraphs. It is a newer discovery explored in the 20^{th} century and is now a concept in discrete mathematics. In this Chapter, we discuss the background of hypergraphs, some basic definitions along with several applications of hypergraphs in general. Also, we compare and contrast graphs and hypergraphs. We see how graphs evolved into hypergraphs. In Chapter 2, we classify the different kinds of linear hypergraphs. They are classified based on uniformity and having cycles. In Chapter 3, we classify the different kinds of non-linear hypergraphs. They are classified based on being balanced and having cycles. In Chapter 4, we explore and present some necessary conditions for the existence of some linear and non-linear cyclic and acyclic hypergraphs. In Chapter 5, we introduce hypergraphs to early college students by presenting simple True/False questions, direct responses, computations along with reasoning questions. Detailed solutions and explanations are also presented. Lastly, in Chapter 6, we conclude our work with some discussions on future research involving the classifications of these hypergraphs which generalize block designs such as Steiner systems.

1.1 Background of Hypergraphs

Hypergraphs became an independent theory and mathematical field in the early 1960s. Hypergraphs were developed in Hungary and France under four main mathematicians. It was first developed by Claude Berge in 1960 in France. The other mathematicians involved included Paul Erdös, László Lovász, Paul Turán. In today's world, applications of hypergraphs include being used for social networks analysis and service-oriented architecture. Since social networking sites have expanded and become more complex, hypergraphs have become useful to show these complex relationships. For example, Facebook's data can now be represented by a hypergraph to show the data sets of someone's close friends, friends, and public. Hypergraphs can be used to help businesses create service-oriented architecture such as application development and application integration.

1.2 Basic Definitions of Graphs and Hypergrpahs

1.2.1 Basic Graphs Definitions

This is a list of basic definitions of Graph Theory found in [18]

- A **graph** G consists of a finite, non-empty set of elements called vertices, denoted by $V(G)$. Graph G a finite family of unordered pairs of elements of $V(G)$ called edges, denoted by $E(G)$.
- $V(G)$ is called the **vertex set** of G. $E(G)$ is called the **edge family** of G.
- The number of edges incident with a vertex v in G is called the **degree** of vertex v.
- The number of edges present within a given graph G is called the **size** of G, also known as $|E(G)|$.
- The number of vertices present within a given graph G is called the **order** of G, denoted buy $|V(G)|$.
- A graph in which there is *at most* one edge joining any given pair of vertices and there are no edges that join a vertex with itself, known as *loops*, is called a **simple graph**.
- A graph is said to be **connected** if for each pair of vertices v, w there is a sequence of vertices $v_0, v_1, v_2, \ldots, v_n$, where $v_0 = w$ and $v_n = v$ such that $v_i v_{i+1}$ is an edge where $0 \leq i \leq n-1$. In laments terms, one can trace every vertex to all other vertices in a graph through a series of edges. This is known as a **path**.

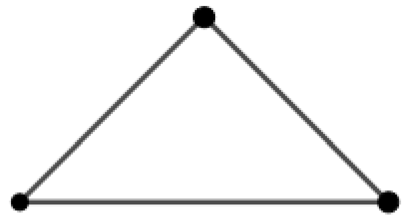


Figure 1.1: Example of Simple Graph

1.2.2 Basic Hypergraphs Definitions

A **hypergraph** is a graph H such that $H = (V, E)$, where V denotes a finite set of *vertices* and E denotes the set of *hyperedges*.

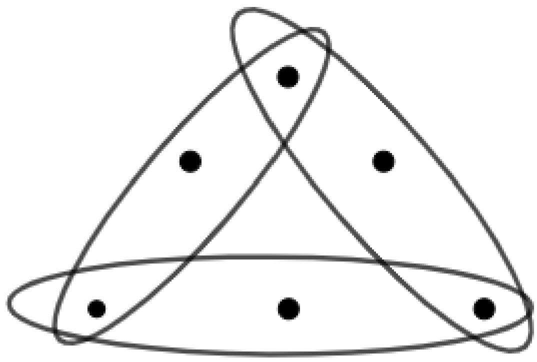


Figure 1.2: Example of Hypergraph

Below are some basic definitions for hypergraphs which can be found in [3].

- A **hyperedge** connects multiple vertices to form a hypergraph. Each hyperedge is a non-empty subset of V.
- A hypergraph H is called **r-uniform**, where r is an integer, if, for each edge $e \in E(H)$, $|e| = r(r \geq 2)$.
- Hypergraphs that are **2-uniform** are considered to be **simple graphs**.
- The **degree** of a vertex v denoted by $d(v)$, is the number of hyperedges that contains v.

- The **length** of a hypergraph is the number of hyperedges it contains.
- A hypergraph H is considered to be **linear** if each pair of hyperedges has at most one vertex in common, otherwise, it is considered to be **non-linear**.
- A hypergraph is H is considered to be **balanced** when the gamma space is a singleton.
- A hypergraph H is considered to be **acyclic** when it is cycle-free.
- A hypergraph is H is considered to be **cyclic** when it has at least one cycle.

1.3 Some Applications of Hypergraphs

Graphs and hypergraphs share many similarities since hypergraphs are generalization of graphs. An edge in a graph is seen as a line. Each pair of vertices creates an edge. A hyperedge is seen as a surface. Each set of vertices creates a hyperedge. In addition, graphs are a subset of hypergraphs. A hypergraph is the higher-order analogue of a graph. Using this type of graph, we are able to model a data set in more ways. Graph theory is limited since it only represents binary connections. Complex systems can be shown using a hypergraph. A 3-uniform hypergraph is the natural way to model the variable structure of a 3SAT problem; 3SAT is one of the most important algorithmic problems in computational complexity theory [11]. Further, Pagerank [14, 15] is a commonly used graph analytic algorithm to find the relative importance of the vertices in a network or search engines. For example, in a social network context we can measure the importance of a user based on the group membership, e.g., owner of a Twitter account with a group of followers would have a bigger influence in the whole network. Among countless applications, hypergraphs have been broadly used as the data model and classifier regularization in machine learning as they are involved in the implementations of image retrieval [6], recommender systems [7] and bioinformatics [8]. For example, clustering is in fact a form of integrating vertices into larger groups from an input hypergraph to compute a rough hypergraph [9].

Chapter 2 Classifying Linear Hypergraphs

In this chapter, we proceed to discuss the first of two big classes of hypergraphs: linear and non-linear. Starting with linear hypergraphs, we divide them into two subclasses: acyclic and cyclic. Each subclass is broken down according to the relationship among the sizes of the hyperedges of its members.

2.1 Linear Hypergraphs

For a hypergraph to be considered linear, each pair of hyperedges will intersect at most one vertex. This is depicted in Figure 2.1. A linear hypergraph can also be:

- Cyclic: at least one cycle in hypergraph
- Acyclic: cycle-free hypergraph
- Uniform: hyperedges contain same number of vertices
- Semi-Uniform: the difference between the size of each pair of hyperedges is at most one.
- Non-Uniform: the difference between the sizes of two or more hyperedges is greater than one.

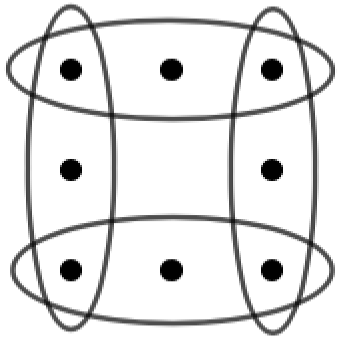


Figure 2.1: Linear Hypergraph

Figure 2.1 is linear because each pair of hyperedges share one vertex.

2.2 Linear Cyclic Hypergraphs

For a hypergraph to be linear and cyclic, it must meet the following requirments:

- Each pair of hyperedges must intersect at only one vertex.
- The hyperedges must create at least one cycle.

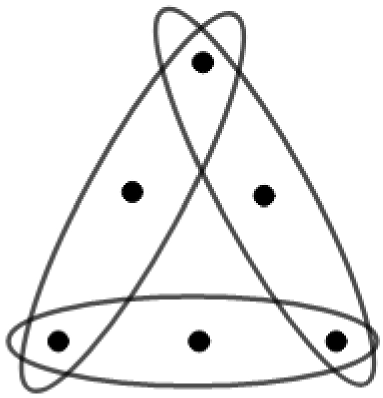


Figure 2.2: Linear Cyclic Hypergraph

Figure 2.2 is linear because each pair of hyperedges share one vertex and there is one cycle.

2.2.1 Linear Uniform Cyclic

For a hypergraph to be linear, uniform, and cyclic it must meet the following requirments:

- Each pair of hyperedges must intersect at most one vertex.
- Every edge family must contain the same number of vertices.
- The hyperedges must create at least one cycle.

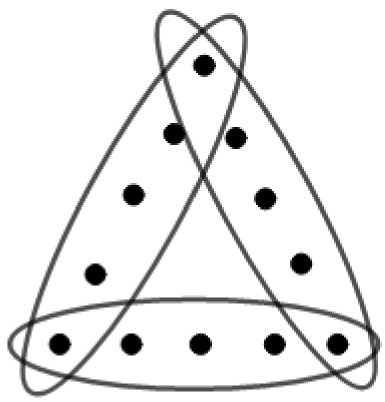


Figure 2.3: Linear 5-Uniform Cyclic Hypergraph

Figure 2.3 is linear because each pair of hyperedges share one vertex, the edge families have five vertices each, and there is one cycle.

2.2.2 Linear Semi-Uniform Cyclic

For a hypergraph to be linear, semi-uniform, and cyclic it must meet the following requirments:

- Each pair of hyperedges must intersect at most one vertex.
- Only one edge family can contain a different number of vertices than the others.
- The hyperedges must create at least one cycle.

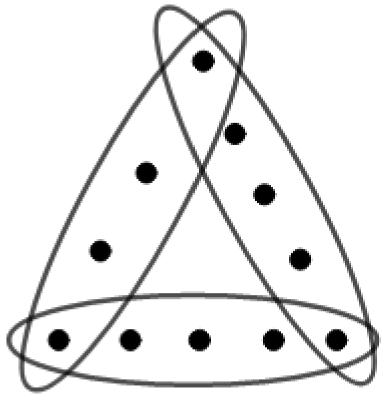


Figure 2.4: Linear Semi-Uniform Cyclic Hypergraph

Figure 2.4 is linear because each pair of hyperedges share one vertex, the edge families have the same number of vertices except for one hyperedge, and there is one cycle.

2.2.3 Linear Non-Uniform Cyclic

For a hypergraph to be linear, non-uniform, and cyclic it must meet the following requirements:

- Each pair of hyperedges must intersect at most one vertex.
- Some hyperedges do not contain the same number of vertices.
- The hyperedges must create at least one cycle.

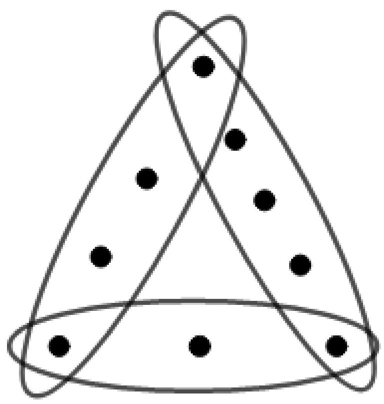


Figure 2.5: Linear Nonuniform Cyclic Hypergraph

Figure 2.5 is linear because each pair of hyperedges share one vertex, the edge families do not have the same number of vertices, and there is one cycle.

2.3 Linear Acyclic Hypergraphs

For a hypergraph to be linear and acyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at most one vertex.
- The hyperedges must not create a cycle in the hypergraph.

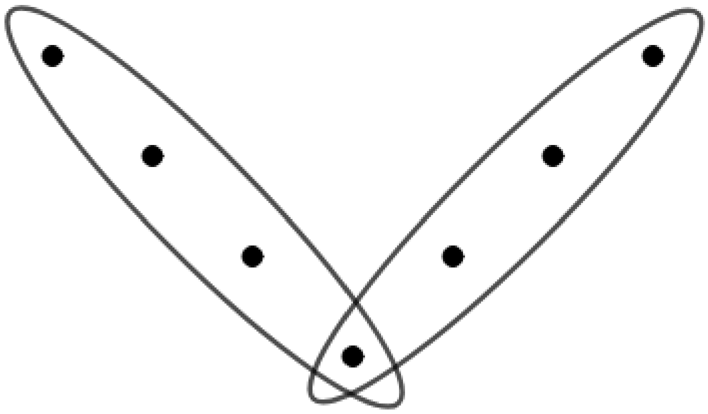


Figure 2.6: Linear Acyclic Hypergraph

Figure 2.6 is linear because each pair of hyperedges share one vertex and there no cycle.

2.3.1 Linear Uniform Acyclic

For a hypergraph to be linear, uniform, and acyclic it must meet the following requirements:

- Each pair of hyperedges must intersect at most one vertex.
- Every edge family must contain the same number of vertices.
- The hyperedges must not create a cycle.

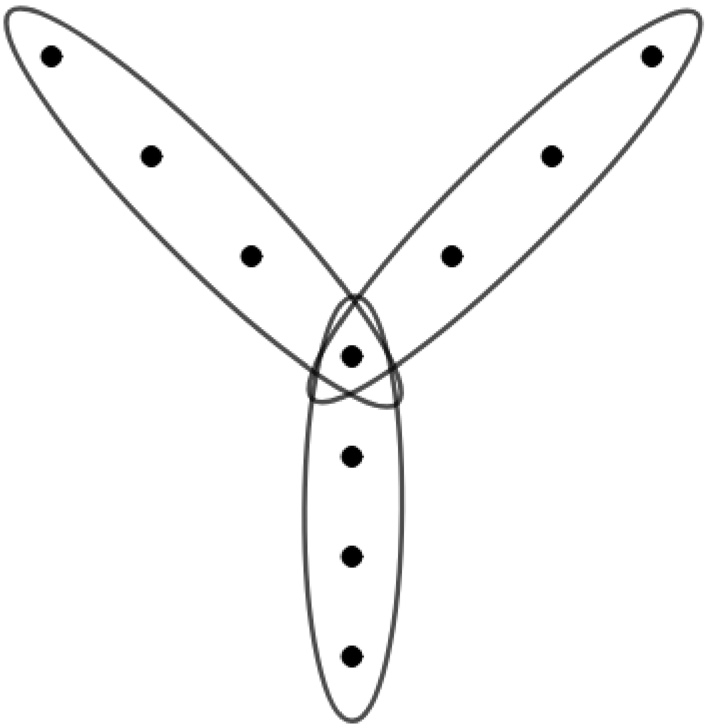


Figure 2.7: Linear 4-Uniform Acyclic Hypergraph

Figure 2.7 is linear because each pair of hyperedges share one vertex, the edge families have four vertices each, and there is no cycle.

2.3.2 Linear Semi-Uniform Acyclic

For a hypergraph to be linear, semi-uniform, and acyclic it must meet the following requirments:

- Each pair of hyperedges must intersect at most one vertex.
- Only one edge family can contain a different number of vertices than the others.
- The hyperedges must not create a cycle.

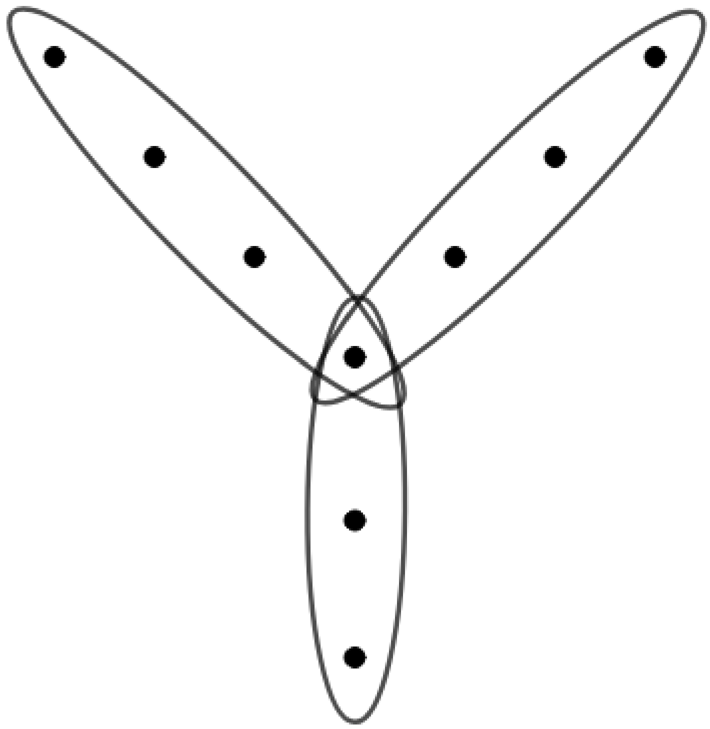


Figure 2.8: Linear Semi-Uniform Acyclic Hypergraph

Figure 2.8 is linear because each pair of hyperedges share one vertex, the edge families have the same number of vertices except for one hyperedge, and there is no cycle.

2.3.3 Linear Non-Uniform Acyclic

For a hypergraph to be linear, non-uniform, and acyclic it must meet the following requirements:

- Each pair of hyperedges must intersect at most one vertex.
- Some hyperedges do not contain the same number of vertices.
- The hyperedges must not create a cycle.

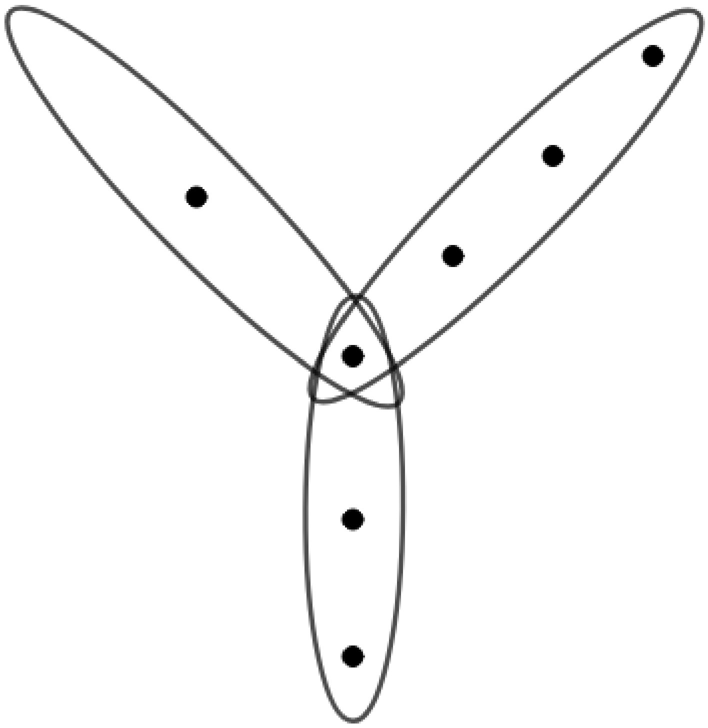


Figure 2.9: Linear Nonuniform Acyclic Hypergraph

Figure 2.9 is linear because each pair of hyperedges share one vertex, the edge families do not have the same number of vertices, and there is no cycle.

Chapter 3 Classifying Non-Linear Hypergraphs

In this chapter, we introduce and classify the least known class of hypergraphs, namely non-linear hypergraphs. For simplicity, the *non-empty intersecting set* of any pair of hyperedges will be referred to as **joint** and we denote its cardinality by some $\gamma \geq 1$ value.

In other words, given a connected hypergraph $H = (V, \mathcal{E})$ we can assume that, for some $e_i, e_j \in \mathcal{E}$, $|e_i \cap e_j| = \gamma$ with $1 \leq \gamma < \max\{|e_i|, |e_j|\}$. It is clear that when some $\gamma > 1$, we are dealing with a non-linear hypergraph. A **Γ-spectrum** of order $r = |\mathcal{E}| - 1$, denoted by $\Gamma = \Gamma(H) =< \gamma_1, \gamma_2, \ldots, \gamma_{|\mathcal{E}|-1} >$, is a sequence of γ values of a given hypergraph H. Further, the set of distinct γ values denoted by Γ^* is called a **Γ-space**. For example, a linear hyperpath on l hyperedges, has a Γ-spectrum of order l, $\Gamma =< 1, 1, \ldots, 1 >$ and a Γ-space $\Gamma^* = \{1\}$. In fact, every linear acyclic hypergraph share the same Γ-spectrum and the same Γ-space, as previously described. However, for non-linear hypergraphs, these two parameters can differ significantly. We have every reason to conjecture that, when paired with a **degree sequence**, one can describe all uniform and non-uniform hypergraphs. Recall, the **degree** of a vertex $v \in V$ is the number of hyperedges that contains the vertex; the minimum and maximum degrees are often denoted by δ and Δ, respectively. For the purpose of this research we shall focus on the two extreme degree cases: $\delta = 1$ and $\Delta = r$, with $r \geq 2$. With these concepts we classify all non-linear hypergraphs into three categories or classes:

1. If $\forall\ (\gamma_i, \gamma_j) \in \Gamma(H)$, $\gamma_i - \gamma_j = 0$, then H is said to be a **balanced hypergraph**. In particular, the special case when $\gamma_i = \gamma_j = 1$, $\forall\ (\gamma_i, \gamma_j) \in \Gamma(H)$, H is a **linear hypergraph** as discussed in the previous chapter. Consequently, the Γ-space is a singleton.

We note here that, every linear hypergraph is balanced but not every balanced hypergraph is linear.

2. If $\forall\ (\gamma_i, \gamma_j) \in \Gamma(H)$, $\gamma_i - \gamma_j = 1$, then H is said to be a **semi-balanced hypergraph**. In general, the Γ-space of every semi-balance hypergraph is a pair although the converse is not true. Also there is no semi-balanced linear hypergraph.

3. If $\exists\ (\gamma_i, \gamma_j) \in \Gamma(H)$, $\gamma_i - \gamma_j > 1$, then H is said to be an **unbalanced hypergraph**.

3.1 Non-Linear Hypergraphs

A hypergraph is considered non-linear if its hyperedges share more than one vertex. An example is shown in Figure 3.1. A non-linear hypergraph can also be:

- Balanced: if $|\Gamma^*| = 1$
- Semi-balanced: if $|\Gamma^*| = 2$
- Unbalanced: if $|\Gamma^*| > 2$
- Cyclic: at least one cycle in hypergraph
- Acyclic: cycle-free hypergraph

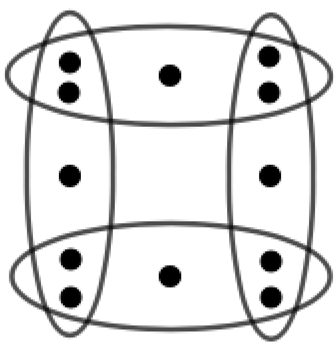


Figure 3.1: Non-Linear Hypergraph

Figure 3.1 is non-linear because each pair of hyperedges share two vertices.

3.2 Non-linear Balanced Hypergraphs

For a hypergraph to be non-linear and balanced, it must meet the following requirements:

- At least one pair of hyperedges intersect at more than one vertex.
- The cardinality of the gamma space is a singleton.

3.2.1 Non-linear Balanced Cyclic

For a hypergraph to be non-linear, balanced, and cyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of gamma space is a singleton.
- The hyperedges must create at least one cycle.

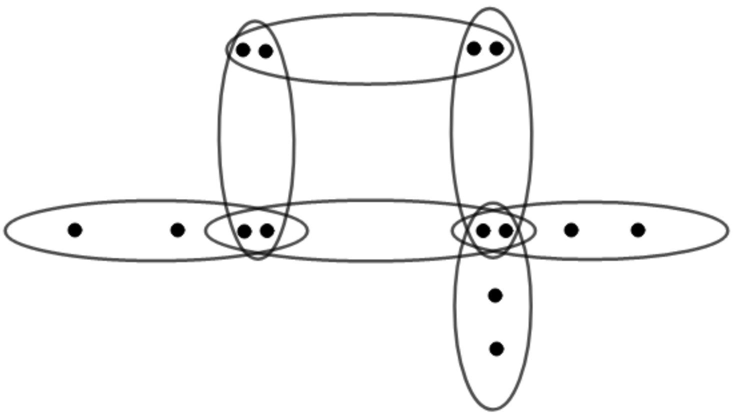


Figure 3.2: Non-Linear Balanced Cyclic Hypergraph

Figure 3.2 is non-linear because there is at least two hyperedges which share more than one vertex. The hypergraph is balanced because $|\Gamma^*| = 1$, i.e., all intersecting hyperedges share exactly the same number of vertices. The hypergraph also contains one cycle, which makes it cyclic. Observe that, the Γ-spectrum, $\Gamma =< 2, 2, 2, 2 >$.

3.2.2 Non-linear Balanced Acyclic

For a hypergraph to be non-linear, balanced, and acyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of gamma space is a singleton.
- The hyperedges must not create a cycle.

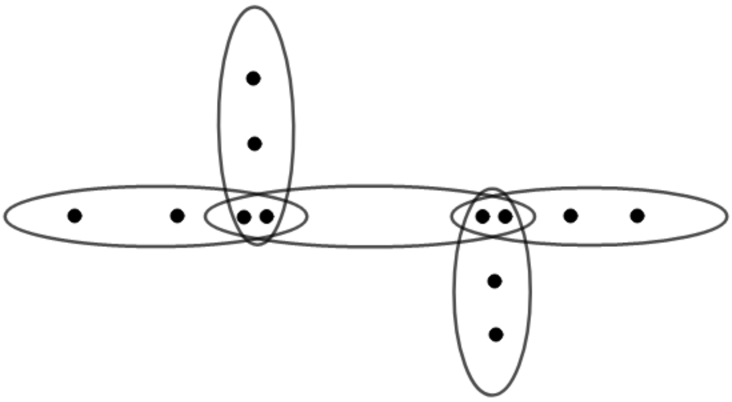


Figure 3.3: Non-Linear Balanced Acyclic Hypergraph

Figure 3.3 is non-linear because the hyperedges intersect at two vertices. The hypergraph is balanced because $|\Gamma^*| = 1$. The hypergraph does not contain a cycle, which makes it acyclic. Observe that, the Γ spectrum, $\Gamma =< 2, 2 >$.

3.3 Non-linear Semi-balanced Hypergraphs

For a hypergraph to be non-linear and semi-balanced, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of the gamma space is two.

3.3.1 Non-linear Semi-balanced Cyclic

For a hypergraph to be non-linear, semi-balanced, and cyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of the gamma space is two.
- The hyperedges must create at least one cycle.

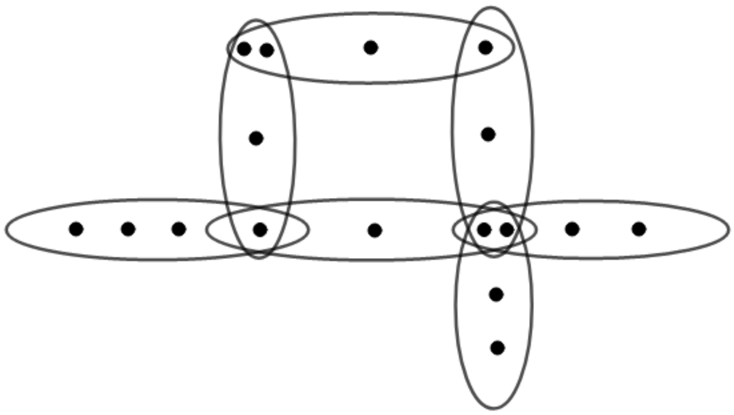


Figure 3.4: Non-Linear Semi-balanced Cyclic Hypergraph

Figure 3.4 is non-linear because the hyperedges intersect at two vertices. The hypergraph is semi-balanced because $|\Gamma^*| = 2$. The hypergraph also contains one cycle, which makes it cyclic. Observe that, the Γ spectrum, $\Gamma =< 2, 1, 1, 2 >$.

3.3.2 Non-linear Semi-balanced Acyclic

For a hypergraph to be non-linear, semi-balanced, and acyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of the gamma space is two.
- The hyperedges must not create a cycle.

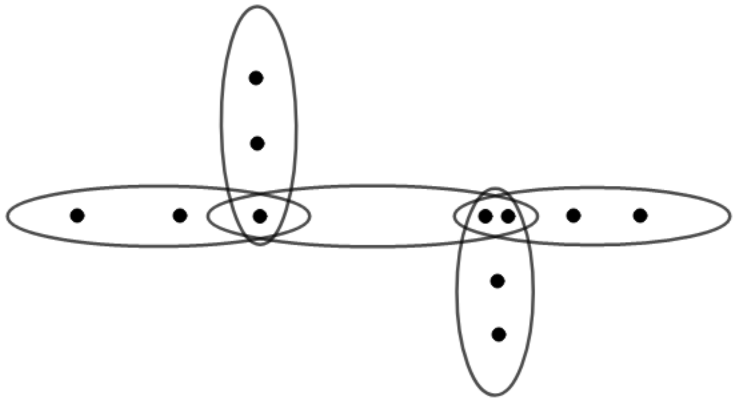


Figure 3.5: Non-Linear Semi-balanced ayclic Hypergraph

Figure 3.5 is non-linear because the hyperedges intersect at two vertices. The hypergraph is semi-balanced because $|\Gamma^*| = 2$. The hypergraph does not contain a cycle, which makes it acyclic. Observe that, the Γ spectrum, $\Gamma =< 1, 2 >$.

3.4 Non-linear Unbalanced Hypergraphs

For a hypergraph to be non-linear and unbalanced, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of the gamma space is greater than two.

3.4.1 Non-linear Unbalanced Cyclic

For a hypergraph to be non-linear, unbalanced, and cyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of the gamma space is greater than two.
- The hyperedges must create at least one cycle.

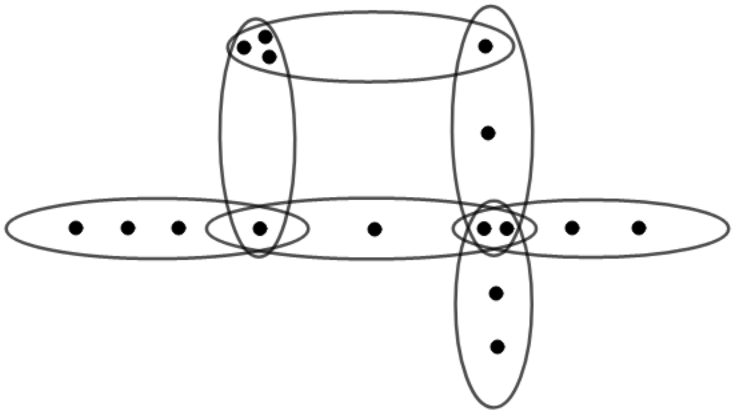


Figure 3.6: Non-Linear Unbalanced Cyclic Hypergraph

Figure 3.6 is non-linear because the hyperedges intersect at two vertices. The hypergraph is unbalanced because $|\Gamma^*| > 2$. The hypergraph also contains one cycle, which makes it cyclic. Observe that, the Γ spectrum, $\Gamma =< 3, 1, 1, 2 >$.

3.4.2 Non-linear Unbalanced Acyclic

For a hypergraph to be non-linear, unbalanced, and acyclic, it must meet the following requirements:

- Each pair of hyperedges must intersect at more than one vertex.
- The cardinality of the gamma space is greater than two.
- The hyperedges must not create a cycle.

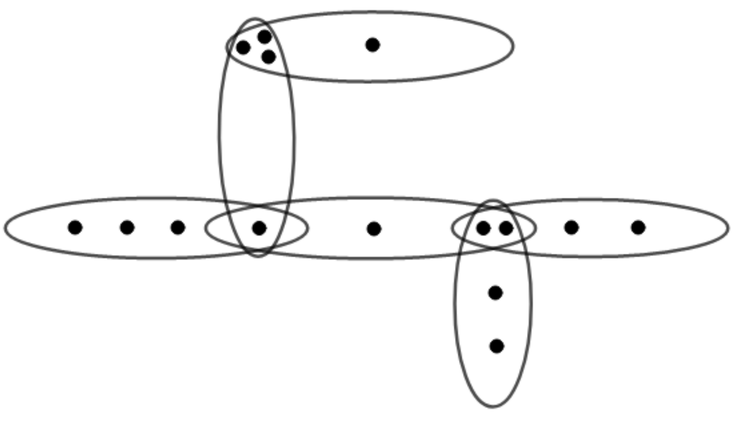


Figure 3.7: Non-Linear Unbalanced Acyclic Hypergraph

Figure 3.7 is non-linear because the hyperedges intersect at two vertices. The hypergraph is unbalanced because $|\Gamma^*| > 2$. The hypergraph does not contain a cycle, which makes it acyclic. Observe that, the Γ spectrum, $\Gamma =< 3, 1, 2 >$.

Chapter 4 On Existence of Hypergraphs

In the previous chapters, we have introduced and classified several hypergraphs. Yet, it is worth noting that, given some constraints on either the number of vertices, the size of the hyperedges, the sizes of the joints, the presence or not of a cyclic hyperpath, many of these hypergraphs may or may not exist. A natural question is under what conditions can such hypergraphs exists? Broadly speaking, hypergraphs are intersecting sets. As such, given any random family of sets, some intersections may not exist, some sizes may be different, etc. Still, if one desires to form a particular family, it becomes very important to know under what conditions is the formation feasible. Such feasibility leads to the notion of the existence of this hypergraph. Classic examples involve **Steiner systems** [17].

4.1 Existence of Some Acyclic Hypergraphs

A *sunflower (hypergraph)* $\mathcal{H}^l = (X, \mathcal{E})$ (also known as a $\Delta-$ system in [5]) with l petals and a core S is a collection of sets $e_1, \ldots, e_l$ such that $e_i \cap e_j = S$ for all $i \neq j$. The elements of the core are called *seeds*. A Venn diagram of these sets would look like a sunflower. Observe that any family of pairwise disjoint sets is a sunflower (with an empty core) and a *hyperstar* is a sunflower with a core of size 1. Figure 4.1 is an example of a $5-$uniform sunflower with a core S of size 3 having e_1, e_2 and e_3 as petals.

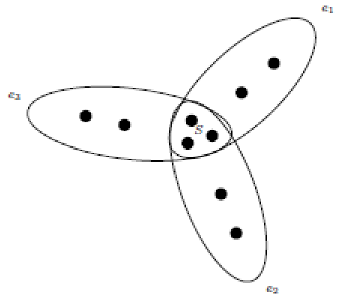


Figure 4.1: A Sunflower

Erdös-Rado sunflower Lemma [5] gives a necessary condition for the existence of a Sunflower given any collection of uniform sets. This condition is a lower bound for the size (or cardinality) of the collection although it is not known if the bound is best possible. We restate the lemma without proof, which is by induction on k.

Sunflower Lemma: *Given any collection of n distinct sets of size k (from a universal set) with $n > k!(l-1)^k$, there is a subcollection of l-sets that forms a sunflower.*

We happen to think that this bound is not best possible. Take for instance, $n = 6$. It is clear that the triples $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$ form a balanced non-linear 3-uniform Sunflower. Yet, $6 = n < l(k) << k!(l-1)^k$, the proposed bound by Erdös and Rado. Here, we propose a necessary condition on the existence of such Sunflowers.

Theorem 4.1.1. *If $\mathscr{H}$ is a k-uniform Sunflower of order m on α seeds, then $m \equiv \alpha \mod (k-\alpha)$.*

Proof. Suppose $\mathscr{H}$ is a k-uniform Sunflower of order m on α seeds with length $l \geq 2$. Then, remove the α seeds from the order m vertices. This leaves $m - \alpha$ elements to be equally distributed among all l-petals. In which case, for each hyperedge, we have $\dfrac{m-\alpha}{l} = k - \alpha$, the numbers of elements not being part of the core. This implies that $m = l(k-\alpha) + \alpha$, for all l. Hence, the proof. □

From Theorem 4.1.1 follows the next corollary.

Corollary 4.1.1.1. *If $\mathscr{H}$ denotes a linear (with $\alpha = 1$) or nonlinear (with $\alpha > 1$) Sunflower of order m with length l on α seeds then $m \geq l(k-\alpha) + \alpha$.*

A second family of Sunflower has recently been introduced by Allagan in [1]. A *transversal* (or *blocking set*) of $\mathscr{F} = \{e_1, \ldots, e_l\}$ is a set which intersects every member of $\mathscr{F}$. A transversal with the least member is often referred to as a *covering set* and its size is called a *covering number* (or *blocking number*). The core S of a sunflower is a transversal and S is a covering set if $|S| = 1$. Let $\mathscr{F} = \{e_1, \ldots, e_l\}$ be a collection of pairwise disjoint sets and we denote by S a transversal of the collection. We call S the core of the collection and its elements are the seeds. The members of the collection will be referred to as petals. In [1], Allagan was first to introduce the following hypergraph $\mathcal{H}^l = (X, \mathcal{E})$ with $\mathcal{E} = S \cup \mathscr{F}$ as a *weak sunflower* with $l-$petals. Figure 4.2 is a representation of a $4-$uniform weak non-linear semi-balanced sunflower with a core S of size 6 having petals e_1, e_2 and e_3.

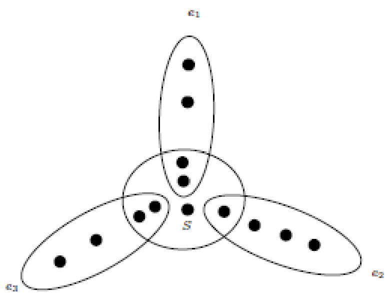


Figure 4.2: $4-$uniform weak non-linear semi-balanced sunflower on 3 petals

Although Erdös-Rado sunflower lemma gives some necessary condition for the existence of a (strong) sunflower, albeit using a weak bound, there is no known condition on the existence of a weak sunflower given a collection of n distinct $k-$uniform sets.

4.2 Existence of Some Cyclic Hypergraphs

A Steiner system with parameters t, k, n, written $S(t, k, n)$, is an n-element set S together with a set of k-element subsets of S (called *blocks* or *hyperedges*) with the property that each t-element subset of S is contained in exactly one block. The particular well-studied case involves Steiner triple systems of order n. A Steiner triple system of order n, denoted by $STS(n)$, is a $(V; \mathscr{B})$ design with n points and each block or hyperedge $B \in \mathscr{B}$ has $|B| = 3$ (also known as triples) such that each pair of distinct elements $x, y \in V$ occur together in exactly one hyperedge of $\mathscr{B}$. For example,

- for $STS(7)$, we have $V = \{1, \ldots, 7\}$ and the family of hyperedges are $B = \{(1,2,3),(1,4,5),(1,6,7),(2,4,6),(2,5,7),(3,4,7),(3,5,6)\}$.

- For $STS(9)$, we have $V = \{1, \ldots, 9\}$ and the family of hyperedges are $B = \{(1,2,3),(4,5,6),(7,8,9),(1,4,7),(2,5,8),(3,6,9),(1,5,9),(2,6,7),(3,4,8),(1,6,8),(2,4,9),(3,5,7)\}$

Steiner triple systems were defined for the first time by W.S.B. Woolhouse (prize question 1733, Lady's and Gentleman's Diary, 1844 [16]) which asked for which positive integers n does a $STS(n)$ exists. This was solved by Rev. T.P. Kirkman, who proved that the following necessary conditions are sufficient.

Theorem 4.2.1. *A $STS(n)$ exists if and only if $n \equiv 3 \pmod 6$*

Because no tuple belongs to two or more blocks, by the necessary condition, it is clear that $STS(7)$ are families of 3-uniform linear cyclic hypergraphs of order 7 with length 7. Also, $STS(9)$ are families of 3-uniform linear cyclic hypergraphs of order 9 with length 12. As such, Steiner systems or block designs, $S(t,k,n)$, are special families of k-uniform linear cyclic hypergraphs of order n. Here, we provide a necessary condition for the existence of any uniform hypergraph.

Theorem 4.2.2. *If $\mathscr{H}$ is a k-uniform linear hypercycle of order m with length l, then $m \equiv l \mod (k-2)$.*

Proof. Suppose $\mathscr{H}$ is a k-uniform linear hypercycle of order m with length l. There are exactly l joints, each contributes 2 vertices to the size of each l hyperedge. Then, remove the l joints from the order m vertices. This leaves $m-l$ elements to be equally distributed among all l-petals. In which case, for each hyperedge, we have $\dfrac{m-l}{l} = k-2$, the numbers of elements of degree 1. This implies that $m = l(k-2)+l$, for all l. Hence, the proof. □

From Theorem 4.2.2 follows the next corollary.

Corollary 4.2.2.1. *If $\mathscr{H}$ denotes a linear cyclic hypergraph of order m with length l then $m \geq l(k-2)+l$.*

Chapter 5 Related Problems to Hypergraph Concepts

5.1 Activities

In this section, we present the reader with many questions, both theoretical and applied. Solutions or answers are also given to each question.

5.1.1 Part I

Match the following terms to the correct definitions.

A. Hypergraph

B. Simple Graph

C. Hyperedge

D. Acyclic Hypergraph

E. Cyclic Hypergraph

F. Linear Hypergraph

G. Non-Linear Hypergraph

H. Uniform Hypergraph

I. Semi-Uniform Hypergraph

J. Non-Uniform Hypergraph

K. Balanced Hypergraph

L. Semi-Balanced Hypergraph

M. Unbalanced Hypergraph

1. ___A hypergraph that has no two edges intersecting at more than one vertex.

2. ___A hypergraph that is 2-uniform.

3. ___A hypergraph is considered to be this when the gamma space is a singleton.

4. ___Connects multiple vertices to form a hypergraph.

5. ___Every edge family contains the same number of vertices.

6. ___A hypergraph that is cycle-free.

7. ___A graph H such that $H = (V, E)$, where V denotes a finite set of *vertices* and E denotes the set of *hyperedges*.

8. ___A hypergraph that has at least one cycle.

9. ___A hypergraph is considered to be this when the gamma space is greater than two.

10. ___A hypergraph that has only one edge family containing a different number of vertices than the others.

11. ___A hypergraph is considered to be this when the gamma space equals two.

12. ___A hypergraph that has hyperedges sharing more than one vertex.

13. ___A hypergraph that has some hyperedges containing a different number of vertices than the others.

5.1.2 Part II

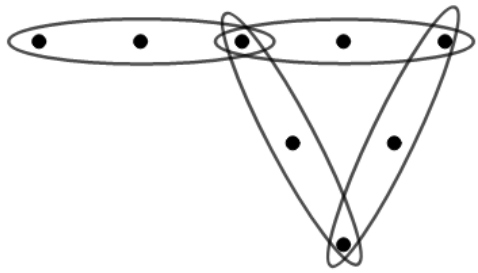
The following are true or false questions. Label each with T for true and F for false.

1. ___Every linear hypergraph is balanced.
2. ___The order of a hypergraph is the number of vertices it contains.
3. ___Every balanced hypergraph is linear.
4. ___Hypergraphs that are 2-uniform are considered to be simple graphs.
5. ___A cyclic hypergraph can have multiple cycles.
6. ___A hypergraph is non-linear if no two edges intersect at more than one vertex.
7. ___The gamma space is used to determine the hypergraph's uniformity.
8. ___The length of a hypergraph is the number of hyperedges it contains.
9. ___The gamma spectrum and gamma spectrum of a hypergraph uses the length of the hypergraph to determine them.
10. ___It is possible to have a cyclic 6-uniform linear hypergraph with order 10 and length 3.

5.1.3 Part III

The following hypergraphs are given. Use the image to determine if the hypergraph is: linear, non-linear, cyclic, acyclic, uniform, semi-uniform, non-uniform, balanced, semi-balanced, or unbalanced. Shade/ select all appropriate boxes.

Hypergraph 1



☐ Linear

☐ Non-Linear

☐ Cyclic

☐ Acyclic

☐ Uniform

☐ Semi-Uniform

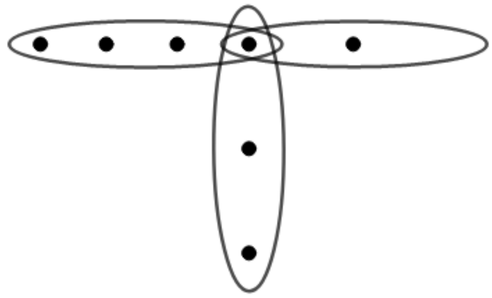
☐ Non-Uniform

☐ Balanced

☐ Semi-Balanced

☐ Unbalanced

Hypergraph 2



☐ Linear

☐ Non-Linear

☐ Cyclic

☐ Acyclic

☐ Uniform

☐ Semi-Uniform

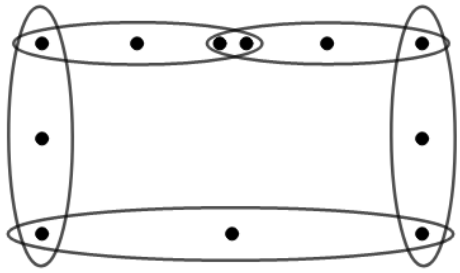
☐ Non-Uniform

☐ Balanced

☐ Semi-Balanced

☐ Unbalanced

Hypergraph 3



☐ Linear

☐ Non-Linear

☐ Cyclic

☐ Acyclic

☐ Uniform

☐ Semi-Uniform

☐ Non-Uniform

☐ Balanced

☐ Semi-Balanced

☐ Unbalanced

5.1.4 Part IV

The following multiple choice questions are given.

1. Which of the following gamma spectrum's, Γ, would be a balanced hypergraph?

   A. $\Gamma =< 2, 3, 1, 2 >$

   B. $\Gamma =< 2, 2, 1 >$

   C. $\Gamma =< 2, 2 >$

   D. $\Gamma =< 1, 3, 1 >$

2. If the gamma spectrum of a hypergraph is $\Gamma =< 3, 3, 3, 3 >$, the gamma space, $|\Gamma^*|$, would be which of the following?

   A. $|\Gamma^*| = 3$

   B. $|\Gamma^*| = 1$

   C. $|\Gamma^*| = 4$

   D. $|\Gamma^*| = 2$

3. What type of hypergraph would it be if it contained two cycles and $\Gamma =< 2, 3, 4, 1 >$?

   A. Linear, Cyclic, Semi-balanced

   B. Non-linear, Acyclic, Unbalanced

   C. Linear, Acyclic, Balanced

   D. Non-linear, Cyclic, Unbalanced

4. What type of hypergraph would it be if it was of length two, $\Gamma^* = 1$, and the order of the hypergraph was ten.

   A. Linear, Acyclic, Balanced, Uniform

   B. Linear, Cyclic, Balanced, Uniform

   C. Non-linear, Cyclic, Unbalanced, Non-uniform

   D. Non-linear, Acyclic, Unbalanced, Semi-uniform

5. If a linear cyclic hypergraph has order 18 and length 3, what is the uniformity of the hypergraph?

   A. 3-uniform

   B. 7-uniform

   C. 5-uniform

   D. not uniform

6. If a linear cyclic hypergraph has order 17 and length 3, what is the uniformity of the hypergraph?

   A. 3-uniform

   B. 7-uniform

   C. 5-uniform

   D. not uniform

7. If a linear acyclic hypergraph has order 17 and length 4, what is the uniformity of the hypergraph?

   A. 4-uniform

   B. 3-uniform

   C. 5-uniform

   D. not uniform

8. What is the order of a linear cyclic hypergraph that is 3-inform and has length of 5?

   A. 10

   B. 5

   C. 7

   D. not a hypergraph

5.1.5 Part V

The following questions are application problems.

1. Is there a linear 4-uniform hypergraph of order 10 and length 3? If there is none, what is the order H that guarantees its existence if:
   (a) H is cyclic?
   (b) H is acyclic?
2. Is there a linear 5-uniform hypergraph of order 10 and length 3? If there is none, what is the order H that guarantees its existence if:
   (a) H is cyclic?
   (b) H is acyclic?
3. Is there a linear 6-uniform hypergraph of order 10 and length 3? If there is none, what is the order H that guarantees its existence if:
   (a) H is cyclic?
   (b) H is acyclic?
4. Is there a linear k-uniform hypergraph of order $m \geq 3$ and length $r \geq 3$? How many are they?
5. How many supervisors are needed to form 4 distinct groups of 3 out of 8 individuals? An individual is a supervisor if and only if he/she belongs to two or more groups.
6. Can you form two groups of five out of ten individuals, if at least one of them has to be the supervisor? An individual is a supervisor if and only if he/she belongs to two or more groups.
7. Can you form a balanced non-linear 5-uniform hypergraph of order 10?
8. Can you form a semi-balanced non-linear hypergraph of order 10?

9. Can you draw a hypergraph $\mathcal{H}$ of order 15 whose Γ-spectrum is $< 1, 1, 3, 2 >$ with hyperedges of size greater or equal to 3? If not, why? Is this spectrum representative of an unbalanced hypergraph? What is the size of Γ^*? Is $\mathcal{H}$ cyclic or acyclic?

5.2 Solutions

In this section, solutions are provided to the given practice problems.

5.2.1 Part I

1. <u>F: Linear</u> A hypergraph that has no two edges intersecting at more than one vertex.
2. <u>B: Simple Graph</u> A hypergraph that is 2-uniform.
3. <u>K: Balanced</u> A hypergraph is considered to be this when the gamma space is a singleton.
4. <u>C: Hyperedge</u> Connects multiple vertices to form a hypergraph.
5. <u>H: Uniform</u> Every edge family contains the same number of vertices.
6. <u>D: Acyclic</u> A hypergraph that is cycle-free.
7. <u>A: Hypergraph</u> A graph H such that $H = (V, E)$, where V denotes a finite set of *vertices* and E denotes the set of *hyperedges*.
8. <u>E: Cyclic</u> A hypergraph that has at least one cycle.
9. <u>M: Unbalanced</u> A hypergraph is considered to be this when the gamma space is greater than two.
10. <u>I: Semi-Uniform</u> A hypergraph that has only one edge family containing a different number of vertices than the others.
11. <u>L: Semi-Balanced</u> A hypergraph is considered to be this when the gamma space equals two.

12. G: Non-Linear A hypergraph that has hyperedges sharing more than one vertex.

13. J: Non-Uniform A hypergraph that has some hyperedges containing a different number of vertices than the others.

5.2.2 Part II

1. _T_ Every linear hypergraph is balanced.

2. _T_ The order of a hypergraph is the number of vertices it contains.

3. _F_ Every balanced hypergraph is linear.

4. _T_ Hypergraphs that are 2-uniform are considered to be simple graphs.

5. _T_ A cyclic hypergraph can have multiple cycles.

6. _F_ A hypergraph is non-linear if no two edges intersect at more than one vertex.

7. _F_ The gamma space is used to determine the hypergraph's uniformity.

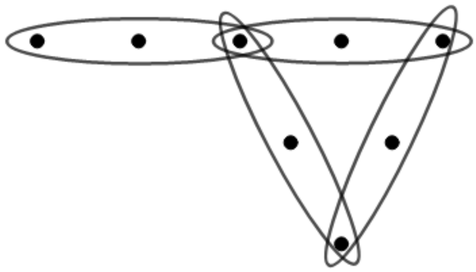
8. _T_ The length of a hypergraph is the number of hyperedges it contains.

9. _F_ The gamma spectrum and gamma spectrum of a hypergraph uses the length of the hypergraph to determine them.

10. _F_ It is possible to have a cyclic 6-uniform linear hypergraph with order 10 and length 3.

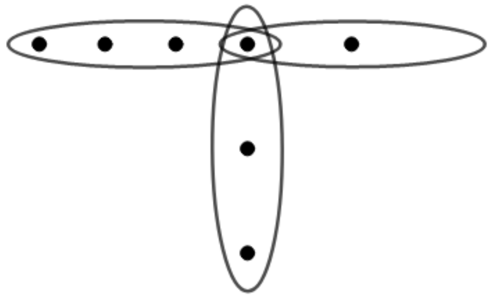
5.2.3 Part III

Hypergraph 1



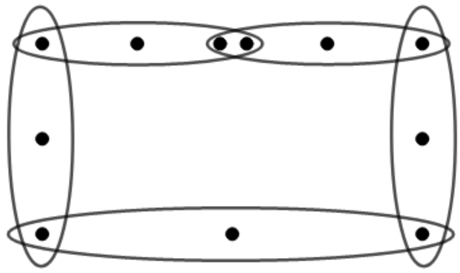
- ■ Linear
- ☐ Non-Linear
- ■ Cyclic
- ☐ Acyclic
- ■ Uniform
- ☐ Semi-Uniform
- ☐ Non-Uniform
- ■ Balanced
- ☐ Semi-Balanced
- ☐ Unbalanced

Hypergraph 2



- ■ Linear
- ☐ Non-Linear
- ☐ Cyclic
- ■ Acyclic
- ☐ Uniform
- ☐ Semi-Uniform
- ■ Non-Uniform
- ■ Balanced
- ☐ Semi-Balanced
- ☐ Unbalanced

Hypergraph 3



- ☐ Linear
- ■ Non-Linear
- ■ Cyclic
- ☐ Acyclic
- ☐ Uniform
- ■ Semi-Uniform
- ☐ Non-Uniform
- ☐ Balanced
- ■ Semi-Balanced
- ☐ Unbalanced

5.2.4 Part IV

1. Which of the following gamma spectrum's, Γ, would be a balanced hypergraph?

   C. $\Gamma =< 2, 2 >$

2. If the gamma spectrum of a hypergraph is $\Gamma =< 3, 3, 3, 3 >$, the gamma space, $|\Gamma^*|$, would be which of the following?

   B. $|\Gamma^*| = 1$

3. What type of hypergraph would it be if it contained two cycles and $\Gamma =< 2, 3, 4, 1 >$?

   D. Non-linear, Cyclic, Unbalanced

4. What type of hypergraph would it be if it was of length two, $\Gamma^* = 1$, and the order of the hypergraph was ten.

   A. Linear, Acyclic, Balanced, Uniform

5. If a linear cyclic hypergraph has order 18 and length 3, what is the uniformity of the hypergraph?

   B. 7-uniform

6. If a linear cyclic hypergraph has order 17 and length 3, what is the uniformity of the hypergraph?

   D. not uniform

7. If a linear acyclic hypergraph has order 17 and length 4, what is the uniformity of the hypergraph?

   C. 5-uniform

8. What is the order of a linear cyclic hypergraph that is 3-inform and has length of 5?

   A. 10

5.2.5 Part V

1. Is there a linear 4-uniform hypergraph of order 10 and length 3? If there is none, what is the order, if possible, of H that guarantees its existence if:

   (a) H is cyclic?

   This hypergraph is not possible to have a combination of length 3 and be 4-uniform.

   (b) H is acyclic?

   This hypergraph is possible having order 10.

2. Is there a linear 5-uniform hypergraph of order 10 and length 3? If there is none, what is the order, if possible, of H that guarantees its existence if:

   (a) H is cyclic?

   This hypergraph is possible having order 10.

(b) H is acyclic?

This hypergraph would need to have order 13.

3. Is there a linear 6-uniform hypergraph of order 10 and length 3? If there is none, what is the order, if possible, of H that guarantees its existence if:

   (a) H is cyclic?

   This hypergraph would need to have order 15.

   (b) H is acyclic?

   This hypergraph would need to have order 16.

4. Is there a linear k-uniform hypergraph of order $m \geq 3$ and length $r \geq 3$? How many are they?

   There are infinitely many for all $k \geq 2$, since $m = r(k-1)$.

5. How many supervisors are needed to form 4 distinct groups of 3 out of 8 individuals? An individual is a supervisor if and only if he/she belongs to two or more groups.

   This question is asking how many vertices are needed to make a linear cyclic hypergraph that is 3-uniform, has length 4, and order 8. By observing Figure 5.1, there would be 4 supervisors needed.

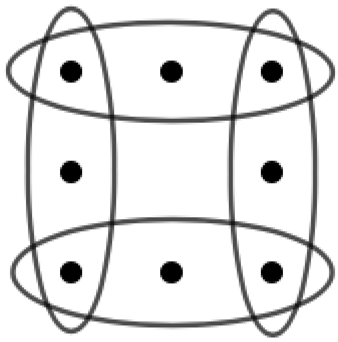


Figure 5.1: Visualization of question 5

6. Can you form two groups of six out of 11 individuals, if at least one of them has to be the supervisor? An individual is a supervisor if and only if he/she belongs to two or more groups.

Yes. Figure 5.2 is an example of a solution to this problem. This question is asking if it is possible to have a hypergraph that is a linear acyclic hypergraph that is 6-uniform, has length 2, and order 11.

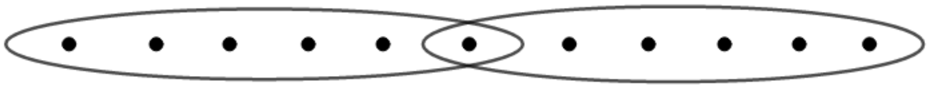


Figure 5.2: Visualization of question 6

7. Can you form a balanced non-linear 5-uniform hypergraph of order 10?

   No, it is not possible to form a balanced non-linear 5-uniform hypergraph of order 10.

8. Can you form a semi-balanced non-linear hypergraph of order 10?

   Yes, it is possible. Figure 5.3 is an example of a solution to this problem. The hypergraph is semi-balanced because $\Gamma =< 2, 1, 1 >$, which then provides us with $|\Gamma^*| = 2$. Since the hypergraph is semi-balanced, it must also be non-linear. In addition, $|V(G)| = 10$.

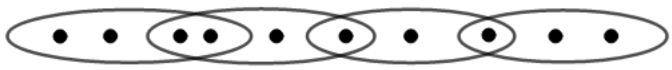


Figure 5.3: Visualization of question 8

9. Can you draw a hypergraph $\mathcal{H}$ of order 15 whose Γ-spectrum is $< 1, 1, 3, 2 >$ with hyperedges of size greater or equal to 3? If not, why? Is this spectrum representative of an unbalanced hypergraph? What is the size of $|\Gamma^*|$? Is $\mathcal{H}$ cyclic or acyclic?

   It is possible to create a hypergraph with these requirements. Figure 5.4 is an example of a solution to this problem. The hypergraph is unbalanced because $\Gamma =< 1, 1, 3, 2 >$, which then provides us with $\Gamma^* > 2$. Since the hypergraph is unbalanced, it must also be non-linear. In addition, $|V(G)| = 15$.

This hypergraph can be created to be either cyclic or acyclic, in this example it is cyclic.

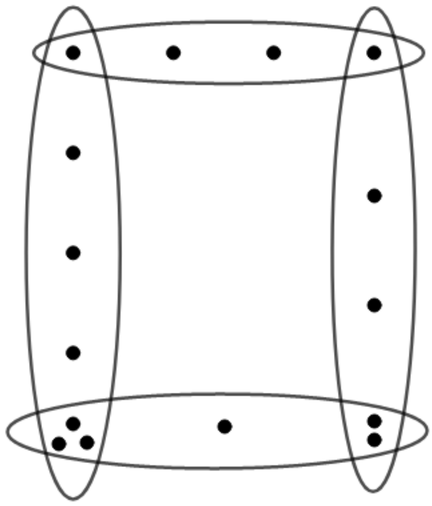


Figure 5.4: Visualization of question 9

Chapter 6 Conclusion and Future Research

In many areas beyond social network, it common to see hypergraphs being used to model chemical reactions, Neural networks, and machine learning. Through this research, we are able to classify several families of hypergraphs. With better classifications come better models use and descriptions. Once classified, these notions can further be used in other hypergraph related researches such as coloring constraints theory, mixed hypergraphs theory, game theory, etc. Our work has been primarily to break hypergraphs into two classes: those that are linear and those that are not, namely non-linear. Each class of such hypergraphs contains sub-classes that are cyclic or acyclic. Further, within each subclass, we separated those that are uniform from those that are not, i.e., non-uniform. Within these groups, we identify members which are balanced vs non-balanced, depending on how varied are the hyperedges' sizes of each hypergraph. As such, we concluded that linear hypergraphs are all balanced, and there may not exist a given member of a hypergraph when linearity, uniformity, or being balanced constraints are applied on a given set of vertices. Although in this research, we have presented several necessary conditions for the existence of some linear and non-linear uniform hypergraphs, more work needs to be done to define their sufficiency conditions and on the existence of several other hypergraph members. These conditions and characterisations will further complement some advanced research studies in design theory that involve Steiner k-systems which are k-uniform hypergraphs that were once referred to as Δ-systems [5] by Erdös and Rado.

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